

SHARP CONSTANT FOR POINCARÉ-TYPE INEQUALITIES IN THE HYPERBOLIC SPACE

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ABSTRACT. In this note, we establish a Poincaré-type inequality on the hyperbolic space \mathbb{H}^n , namely

$$\|u\|_p \leq C(n, m, p) \|\nabla_g^m u\|_p$$

for any $u \in W^{m,p}(\mathbb{H}^n)$. We prove that the sharp constant $C(n, m, p)$ for the above inequality is

$$C(n, m, p) = \begin{cases} (pp'/(n-1)^2)^{m/2} & \text{if } m \text{ is even,} \\ (p/(n-1)) (pp'/(n-1)^2)^{(m-1)/2} & \text{if } m \text{ is odd,} \end{cases}$$

with $p' = p/(p-1)$ and this sharp constant is never achieved in $W^{m,p}(\mathbb{H}^n)$. This generalizes some known results for the case $p = 2$.

1. INTRODUCTION

Given a bounded, connected domain $\Omega \subset \mathbb{R}^n$ with Lipschitz boundary $\partial\Omega$, the classical Poincaré inequality with a sharp constant $C(p, \Omega)$ states that

$$\int_{\Omega} |u|^p dx \leq C(p, \Omega) \int_{\Omega} |\nabla u|^p dx \quad (1.1)$$

for “suitable” function u (usually in the Sobolev space $W^{1,p}(\Omega)$) with vanishing mean value on Ω . Without assuming the vanishing mean value on Ω , the classical Poincaré inequality reads as

$$\int_{\Omega} |u - \bar{u}|^p dx \leq C(p, \Omega) \int_{\Omega} |\nabla u|^p dx \quad (1.2)$$

where $\bar{u} = (1/|\Omega|) \int_{\Omega} u dx$ denotes the mean value (or average) of u over Ω . Inequality (1.1) usually holds for $1 \leq p < +\infty$ under very general assumptions on Ω , for example, it holds for domains satisfying the so-called “segment property” or “cone property”; see [Agm65, LL01]. An interesting question is that how the constant $C(p, \Omega)$ depends on the domain Ω ?

For $p = 2$ and $n = 3$, Steklov [Ste96] showed that the constant $C(2, \Omega)$, when $\partial\Omega$ is piecewise smooth, must equal $1/\lambda_1$ where λ_1 is the first, non-zero eigenvalue of the following Neumann boundary condition problem

$$\begin{cases} -\Delta u = \lambda u & \text{in } \Omega, \\ \partial_{\vec{n}} u = 0 & \text{on } \partial\Omega. \end{cases}$$

Here \vec{n} is the exterior unit normal. A similar result was also obtained by Steklov [Ste97] for the Dirichlet boundary condition problem

$$\begin{cases} -\Delta u = \lambda u & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega. \end{cases}$$

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Based on these fundamental results, a few results for the sharp constant $C(2, \Omega)$ are known; for example, the sharp constant $C(2, B(0, 1))$ for the unit ball in \mathbb{R}^3 is $1/j_{1,1}$ where $j_{1,1}$ is the first positive zero of the Bessel function J_1 ; see [KN15, Subsection 2.2] and [NR15]. For convex domains $\Omega \subset \mathbb{R}^n$ with diameter d , in a beautiful work by Payne and Weinberger [PW60], the authors showed that (1.1) for $p = 2$ can be obtained from weighted Poincaré inequalities in one dimension. As a consequence of this, they proved that $C(2, \Omega) = d/\pi$. A similar argument applied to the case $p = 1$ gives $C(1, \Omega) = d/2$; see [AD04].

Poincaré inequalities for punctured domains was also studied in [LSY03]. For a general domain Ω and arbitrary p , determining the Poincaré constant $C(p, \Omega)$ is a hard task since the value $C(p, \Omega)$ depends on p and the geometry of the domain Ω .

In this note, we consider (1.1) for the hyperbolic space \mathbb{H}^n with $n \geq 2$. The motivation of writing this note goes back to a recent higher order Poincaré-type inequality on \mathbb{H}^n established by Karmakar and Sandeep in [KS16] and subsequently by a few works such that [BG15, BGG15]; for interested readers, we refer to [MS08, Tat01] for further details and related issues. To go further, let us briefly recall the definition of the space \mathbb{H}^n .

The hyperbolic space \mathbb{H}^n with $n \geq 2$ is a complete and simply connected Riemannian manifold having constant sectional curvature equal to -1 . There is a number of models for \mathbb{H}^n , however, the most important models are the half-space model, the ball model, and the hyperboloid (or Lorentz) model. In this note, we are interested in the ball model since this model is especially useful for questions involving rotational symmetry.

Given $n \geq 2$, we denote by B_n the open unit ball in \mathbb{R}^n . Clearly, B_n can be endowed with the following Riemannian metric

$$g(x) = \left(\frac{2}{1 - |x|^2} \right)^2 dx \otimes dx,$$

which is then called the ball model of the hyperbolic space \mathbb{H}^n . In local coordinates, we have $g_{ij} = (2/(1 - |x|^2))^2 \delta_{ij}$ and $g^{ij} = ((1 - |x|^2)/2)^2 \delta^{ij}$. Clearly, one can think that g is conformal to dx^2 with the conformal factor $\ln(2/(1 - |x|^2))$. Then, it is well-known that volume element of \mathbb{H}^n is given by

$$dV_g(x) = \left(\frac{2}{1 - |x|^2} \right)^n dx,$$

where dx denotes the Lebesgue measure in \mathbb{R}^n . Let $d(0, x)$ denote the hyperbolic distance between the origin and the point x . In the ball model, it is well-known that $d(0, x) = \ln((1 + |x|)/(1 - |x|))$ for arbitrary $x \in B_n$. In this new context, we still use ∇ and Δ to denote the Euclidean gradient and Laplacian as well as $\langle \cdot, \cdot \rangle$ to denote the standard inner product in \mathbb{R}^n . Then, in terms of ∇ , Δ , and $\langle \cdot, \cdot \rangle$, with respect to the hyperbolic metric g , the hyperbolic gradient ∇_g , whose local coordinates is $g^{ij} \partial_j$, and the Laplacian-Beltrami operator Δ_g , defined to be $\text{div}_g(\nabla \cdot)$, are given by

$$\nabla_g = \left(\frac{1 - |x|^2}{2} \right)^2 \nabla, \quad \Delta_g = \left(\frac{1 - |x|^2}{2} \right)^2 \Delta + (n - 2) \left(\frac{1 - |x|^2}{2} \right)^2 \langle x, \nabla \rangle.$$

For higher order derivatives, we shall adopt the following convention

$$\nabla_g^m \cdot = \begin{cases} \Delta_g^{m/2} \cdot & \text{if } m \text{ is even,} \\ \nabla_g(\Delta_g^{(m-1)/2} \cdot) & \text{if } m \text{ is odd.} \end{cases}$$

Then the norm of ∇_g^m being calculated with respect to the metric g is understood as follows

$$|\nabla_g^m \cdot|_g = \begin{cases} |\nabla_g^m \cdot| & \text{if } m \text{ is even,} \\ \langle \nabla_g^m \cdot, \nabla_g^m \cdot \rangle_g^{1/2} & \text{if } m \text{ is odd.} \end{cases}$$

For simplicity, we write $|\nabla_g^m \cdot|$ instead of $|\nabla_g^m \cdot|_g$ if no confusion occurs.

Given a function f on \mathbb{H}^n , we denote $\|f\|_p = (\int_{\mathbb{H}^n} |f|^p dV_g)^{1/p}$ and $\|\nabla_g^m f\|_p = \| |\nabla_g^m f|_g \|_p$, for each $1 \leq p < \infty$ and integer $m \geq 1$. We use $W^{m,p}(\mathbb{H}^n)$ to denote the Sobolev space of order m in \mathbb{H}^n . In [KS16], the authors prove the following higher order Poincaré inequality

$$\|\nabla_g^l u\|_2 \leq \left(\frac{2}{n-1}\right)^{m-l} \|\nabla_g^m u\|_2 \quad (1.3)$$

for all $u \in W^{m,2}(\mathbb{H}^n)$. In view of (1.3), one can ask: *Whether the constant $(2/(n-1))^{m-l}$ is sharp and do we have a similar inequality for L^p -norm?* We notice that it was claimed in [BG15] that the constant $(2/(n-1))^{m-l}$ in (1.3) is sharp; however, we have not found any proof of this yet. In this note, we seek for an answer for the above question.

In order to state our results, for each number $1 < p < +\infty$, let us first denote the following constant

$$C(n, m, p) = \begin{cases} (pp'/(n-1)^2)^{m/2} & \text{if } m \text{ is even,} \\ (p/(n-1)) (pp'/(n-1)^2)^{(m-1)/2} & \text{if } m \text{ is odd,} \end{cases} \quad (1.4)$$

with $p' = p/(p-1)$. Clearly when $p = 2$ and hence $p' = 2$, we obtain $C(n, m, 2) = (2/(n-1))^m$. In this note, our first result is the following.

Theorem 1.1. *Given $p > 1$, then the following inequality holds*

$$\|u\|_p \leq C(n, m, p) \|\nabla_g^m u\|_p \quad (1.5)$$

for $u \in W^{m,p}(\mathbb{H}^n)$. Moreover, the constant $C(n, m, p)$ is sharp and is never achieved in $W^{m,p}(\mathbb{H}^n)$.

As a consequence of Theorem 1.1, we know that the sharp constant $C(3, 1, 2)$ is $1/2$ which is not $1/j_{1,1}$ as in the Euclidean case. Let us now go back to (1.3). By making use of Theorem 1.1 above, we obtain the following corollary, which generalizes (1.3).

Corollary 1.2. *Given $p > 1$, then the following inequality holds*

$$\|\nabla_g^l u\|_p \leq C(n, m-l, p) \|\nabla_g^m u\|_p \quad (1.6)$$

for $u \in W^{m,p}(\mathbb{H}^n)$. Moreover, the constant $C(n, m-l, p)$ is sharp and is never achieved in $W^{m,p}(\mathbb{H}^n)$.

As a special case of Corollary (1.2), we conclude that the constant $(2/(n-1))^{m-l}$ in (1.3) is sharp. In view of the results in [BG15], it would be nice, since the sharp constant is never achieved, if there is an analogue of (1.5) with reminders. We leave this topic for interested readers.

2. PROOFS

In this section, we prove Theorem 1.1. Our proof basically consists of two main parts. In the first part, we prove (1.5). Then in the second part, we show that the constant $C(n, m, p)$ is sharp. We start with the first part.

2.1. Proof of (1.5). It is now known that the symmetrization argument works well in the setting of hyperbolic spaces. It is not only the key tool in the proof of several important inequalities such as Adams-Moser-Trudinger in \mathbb{H}^n but also a key tool in the present proof. Let us now recall some facts about the rearrangement in the hyperbolic spaces. Let the function $f : \mathbb{H}^n \rightarrow \mathbb{R}$ be such that

$$|\{x \in \mathbb{H}^n : |f(x)| > t\}| = \int_{\{x \in \mathbb{H}^n : |f(x)| > t\}} dV_g < +\infty$$

for every $t > 0$. Its *distribution function* is defined by

$$\mu_f(t) = |\{x \in \mathbb{H}^n : |f(x)| > t\}|.$$

Then its *decreasing rearrangement* f^* is defined by

$$f^*(t) = \sup\{s > 0 : \mu_f(s) > t\}.$$

Since f^* is non-increasing, the maximal function f^{**} of f^* is defined by

$$f^{**}(s) = \frac{1}{s} \int_0^s f^*(t) dt.$$

It is well-known for any $p \in (1, \infty)$ that

$$\left(\int_0^\infty f^{**}(s)^p ds \right)^{1/p} \leq p' \left(\int_0^\infty f^*(s)^p ds \right)^{1/p}. \quad (2.1)$$

Now, we define $f^\sharp : \mathbb{H}^n \rightarrow \mathbb{R}$ by

$$f^\sharp(x) = f^*(|B(0, d(0, x))|),$$

where $B(0, d(0, x))$ and $|B(0, d(0, x))|$ denote the ball centered at the origin 0 with radius $d(0, x)$ in the hyperbolic space and its hyperbolic volume, respectively. Then for any continuous increasing function $\Phi : [0, +\infty) \rightarrow [0, +\infty)$ we have

$$\int_{\mathbb{H}^n} \Phi(|f|) dV_g = \int_{\mathbb{H}^n} \Phi(f^\sharp) dV_g. \quad (2.2)$$

Moreover, the Polya-Szegö principle conclude that

$$\int_{\mathbb{H}^n} |\nabla_g f^\sharp|^p dV_g \leq \int_{\mathbb{H}^n} |\nabla_g f|^p dV_g$$

for any function $f : \mathbb{H}^n \rightarrow \mathbb{R}$. Now we define a function on $[0, +\infty)$ as follows

$$\Phi(s) = n\omega_n \int_0^s (\sinh r)^{n-1} dr, \quad s \geq 0.$$

Clearly, Φ is a continuous and strictly increasing function from $[0, +\infty)$ to $[0, +\infty)$. Let F denote the inverse function of Φ , it is not hard to verify that F is a continuous, strictly increasing function and satisfies

$$s = n\omega_n \int_0^{F(s)} (\sinh r)^{n-1} dr \quad (2.3)$$

for any $s \geq 0$. Depending on m and for clarity, we divide this part into several small steps as follows.

2.1.1. *The case $m = 1$.* Let $u \in W^{1,p}(\mathbb{H}^n)$ arbitrary. Upon normalization, if necessary, we can assume that $\|\nabla_g u\|_p = 1$. Then by the Polya–Szegő principle we know that $\|\nabla_g u^\sharp\|_p \leq 1$. Recall, by definition, that $u^\sharp(x) = u^*(|B(0, d(0, x))|)$. Let μ_u denote the distribution function of u . For $t > 0$, let $\rho(t)$ denote the radius of the ball having hyperbolic volume $\mu_u(t)$. Then, we have

$$\mu_u(t) = \int_{B(0, \rho(t))} dV_g = n\omega_n \int_0^{\rho(t)} (\sinh s)^{n-1} ds.$$

From this and the definition of the function F , it is easy to check that

$$\rho(t) = F(\mu_u(t)).$$

We now define

$$\varphi(s) = (n\omega_n)^{-p/(p-1)} \int_s^{+\infty} (\sinh F(t))^{-p(n-1)/(p-1)} dt \quad (2.4)$$

and choose

$$g(\varphi(s)) = u^*(s). \quad (2.5)$$

Clearly the function φ is decreasing with

$$-\varphi'(s) = (n\omega_n (\sinh F(s))^{n-1})^{-p/(p-1)}.$$

Concerning to the function g , it is increasing and

$$\int_0^{+\infty} (g'(s))^p ds = \int_{\mathbb{H}^n} |\nabla_g u^\sharp|^p dV_g \leq 1.$$

Denote $\underline{g} = (g')^*$ the decreasing rearrangement of g' on $(0, \infty)$ and set

$$f(s) = \int_0^{\varphi(s)} \underline{g}(t) dt.$$

We have $f(s) \geq u^*(s)$ and

$$\int_0^{+\infty} \underline{g}(s)^p ds = \int_0^{+\infty} (g'(s))^p ds \leq 1.$$

Via integration by parts, we have for any $0 < a < b < +\infty$

$$\begin{aligned} \int_a^b f(s)^p ds &= -p \int_a^b s \varphi'(s) \underline{g}(\varphi(s)) f(s)^{p-1} ds \\ &\quad + b \left(\int_0^{\varphi(b)} \underline{g}(s) ds \right)^p - a \left(\int_0^{\varphi(a)} \underline{g}(s) ds \right)^p. \end{aligned} \quad (2.6)$$

Next we show that

$$\lim_{a \rightarrow 0} a \left(\int_0^{\varphi(a)} \underline{g}(s) ds \right)^p = \lim_{b \rightarrow +\infty} b \left(\int_0^{\varphi(b)} \underline{g}(s) ds \right)^p = 0. \quad (2.7)$$

Indeed, for any $\varepsilon > 0$, there is $R > 0$ such that $\int_R^{+\infty} \underline{g}(s)^p ds < \varepsilon^p$, take s_0 such that $\varphi(s_0) = R$, for $0 < a < s_0$ we have

$$\begin{aligned} \int_0^{\varphi(a)} \underline{g}(s) ds &= \int_0^{\varphi(s_0)} \underline{g}(s) ds + \int_{\varphi(s_0)}^{\varphi(a)} \underline{g}(s) ds \\ &\leq \int_0^{\varphi(s_0)} \underline{g}(s) ds + \left(\int_{\varphi(s_0)}^{\varphi(a)} \underline{g}(s)^p ds \right)^{1/p} (\varphi(a) - \varphi(s_0))^{(p-1)/p} \\ &\leq \int_0^{\varphi(s_0)} \underline{g}(s) ds + \varepsilon (\varphi(a) - \varphi(s_0))^{(p-1)/p}. \end{aligned}$$

Since $n\omega_n(\sinh F(s))^{n-1} \geq (n-1)s$ for all $s > 0$, we conclude that

$$\begin{aligned}\varphi(a) - \varphi(s_0) &\leq \int_a^{s_0} ((n-1)s)^{-p/(p-1)} ds \\ &= (n-1)^{-p/(p-1)}(p-1)(a^{-1/(p-1)} - s_0^{-1/(p-1)}).\end{aligned}$$

Therefore we get

$$\begin{aligned}\limsup_{a \rightarrow 0} a \left(\int_0^{\varphi(a)} \underline{g}(s) ds \right)^p &\leq \limsup_{a \rightarrow 0} a \left(\int_0^{\varphi(s_0)} \underline{g}(s) ds + \varepsilon(\varphi(a) - \varphi(s_0))^{(p-1)/p} \right)^p \\ &= \limsup_{a \rightarrow 0} \left[a \varepsilon^p (\varphi(a) - \varphi(s_0))^{p-1} \right] \\ &\leq (n-1)^{-p}(p-1)^{p-1} \varepsilon^p \\ &\quad \times \limsup_{a \rightarrow 0} a (a^{-1/(p-1)} - s_0^{-1/(p-1)})^{p-1} \\ &= (n-1)^{-p}(p-1)^{p-1} \varepsilon^p.\end{aligned}$$

Since $\varepsilon > 0$ is arbitrary, we get

$$\limsup_{a \rightarrow 0} a \left(\int_0^{\varphi(a)} \underline{g}(s) ds \right)^p = 0.$$

The second limit in (2.7) follows from the Hölder inequality. Indeed, first we have

$$\left(\int_0^{\varphi(b)} \underline{g}(s) ds \right)^p \leq \varphi(b)^{p-1} \int_0^{\varphi(b)} \underline{g}(s)^p ds.$$

Observe that

$$\begin{aligned}\varphi(b) &\leq (n-1)^{-p/(p-1)} \int_b^{+\infty} s^{-p/(p-1)} ds \\ &\leq (n-1)^{-p/(p-1)}(p-1)b^{-1/(p-1)},\end{aligned}\tag{2.8}$$

which helps us to obtain

$$b \left(\int_0^{\varphi(b)} \underline{g}(s) ds \right)^p \leq (n-1)^{-p}(p-1)^{p-1} \int_0^{\varphi(b)} \underline{g}(s)^p ds.$$

From this the conclusion follows by the fact $\lim_{b \rightarrow 0} \int_0^{\varphi(b)} \underline{g}(s)^p ds = 0$ since $\varphi(b)$ tends to 0 as b tends to 0. Thanks to $\varphi' \leq 0$, we can denote

$$h(s) = \underline{g}(\varphi(s))(-\varphi'(s))^{1/p}.$$

Clearly, $\int_0^{+\infty} h(s)^p ds \leq 1$. Making use of the Hölder inequality and (2.6), we can estimate $\int_a^b f(s)^p ds$ as follows

$$\begin{aligned}\int_a^b f(s)^p ds &\leq p \left(\int_a^b (-\varphi'(s) s \underline{g}(\varphi(s)))^p ds \right)^{1/p} \left(\int_a^b f(s)^p ds \right)^{(p-1)/p} \\ &\quad + b \left(\int_0^{\varphi(b)} \underline{g}(s) ds \right)^p - a \left(\int_0^{\varphi(a)} \underline{g}(s) ds \right)^p.\end{aligned}$$

Dividing both sides by $(\int_a^b f(s)^p ds)^{(p-1)/p}$, then letting $a \searrow 0$ and $b \nearrow +\infty$ and using (2.7), we obtain

$$\left(\int_0^{+\infty} f(s)^p ds \right)^{1/p} \leq p \left(\int_a^b [-\varphi'(s) s \underline{g}(\varphi(s))]^p ds \right)^{1/p}.\tag{2.9}$$

Note that the inequality $n\omega_n(\sinh F(s))^{n-1} > (n-1)s$ for any $s > 0$ and the definition of φ imply that

$$(-\varphi'(s))^{(p-1)/p} s < (n-1)^{-1}, \quad \forall s > 0.$$

Combining the latter inequality and (2.9), we obtain

$$\left(\int_0^{+\infty} f(s)^p ds \right)^{1/p} < \frac{p}{n-1} \left(\int_0^{+\infty} h(s)^p ds \right)^{1/p} \leq \frac{p}{n-1},$$

Since $u^* \leq f$, we have

$$\left(\int_{\mathbb{H}^n} |u|^p dV_g \right)^{1/p} = \left(\int_0^{+\infty} (u^*(s))^p ds \right)^{1/p} \leq \left(\int_0^{+\infty} f(s)^p ds \right)^{1/p} < \frac{p}{n-1},$$

for any function $u \in W^{1,p}(\mathbb{H}^n)$ with $\|\nabla_g u\|_p = 1$. This proves (1.5) for the case $m = 1$ and also shows that the constant $C(n, 1, p)$ is not achieved.

2.1.2. *The case $m = 2$.* For any function $u \in W^{2,p}(\mathbb{H}^n)$ such that $\|\Delta_g u\|_p = 1$, denote $f = -\Delta_g u$. It was proved in [NN16] that

$$u^*(s) \leq \int_s^{+\infty} \frac{tf^{**}(t)}{[n\omega_n(\sinh F(t))^{n-1}]^2} dt =: h(s), \quad \forall s > 0.$$

As in the case $m = 1$, we can easily prove that

$$\lim_{s \rightarrow 0^+} sh(s)^p = \lim_{s \rightarrow +\infty} sh(s)^p = 0.$$

For any $b > a > 0$, using integration by parts and the Hölder inequality imply that

$$\begin{aligned} \int_a^b h(s)^p ds &= p \int_a^b h(s)^{p-1} \frac{s^2 f^{**}(s)}{[n\omega_n(\sinh F(s))^{n-1}]^2} ds + bh(b)^p - ah(a)^p \\ &\leq p \left(\int_a^b h(s)^p ds \right)^{(p-1)/p} \left(\int_a^b \left[\frac{s^2 f^{**}(s)}{[n\omega_n(\sinh F(s))^{n-1}]^2} \right]^p ds \right)^{1/p} \\ &\quad + bh(b)^p - ah(a)^p. \end{aligned}$$

Dividing both sides by $\left(\int_a^b h(s)^p ds \right)^{1/p}$ and letting $a \searrow 0$ and $b \nearrow +\infty$, we obtain

$$\left(\int_0^{+\infty} h(s)^p ds \right)^{1/p} \leq p \left(\int_0^{+\infty} \left[\frac{s^2 f^{**}(s)}{[n\omega_n(\sinh F(s))^{n-1}]^2} \right]^p ds \right)^{1/p}. \quad (2.10)$$

Using the inequalities $n\omega_n(\sinh F(s))^{n-1} > (n-1)s$ for any $s > 0$, (2.1) and (2.10), we have

$$\left(\int_0^{+\infty} h(s)^p ds \right)^{1/p} < \frac{pp'}{(n-1)^2} \left(\int_0^{+\infty} f^*(s)^p ds \right)^{1/p} \leq \frac{pp'}{(n-1)^2}.$$

Since $u^* \leq h$, we then obtain

$$\left(\int_{\mathbb{H}^n} |u|^p dV_g \right)^{1/p} = \left(\int_0^{+\infty} (u^*(s))^p ds \right)^{1/p} \leq \left(\int_0^{+\infty} h(s)^p ds \right)^{1/p} < \frac{pp'}{(n-1)^2},$$

for any function $u \in W^{2,p}(\mathbb{H}^n)$ with $\|\Delta_g u\|_p = 1$. This proves (1.5) for the case $m = 2$ and also shows that the constant $C(n, 2, p)$ is not achieved.

2.1.3. *The case $m > 2$.* In this scenario, we have two possible cases:

Case 1. Suppose that $m = 2k$ is even. Clearly, this case follows from the case $m = 2$ by repeating k times as follows

$$\begin{aligned} \|u\|_p &\leq \frac{pp'}{(n-1)^2} \|\Delta_g u\|_p \leq \left(\frac{pp'}{(n-1)^2} \right)^2 \|\Delta_g^2 u\|_p \\ &\leq \dots \leq \left(\frac{pp'}{(n-1)^2} \right)^k \|\Delta_g^k u\|_p. \end{aligned}$$

Case 2. Suppose that $m = 2k + 1$ is odd. This case can also be derived from the cases $m = 1$ and $m = 2$ as the following

$$\begin{aligned} \|u\|_p &\leq \frac{p}{n-1} \|\nabla_g u\|_p \leq \frac{p}{n-1} \frac{pp'}{(n-1)^2} \|\nabla_g(\Delta_g u)\|_p \\ &\leq \dots \leq \frac{p}{n-1} \left(\frac{pp'}{(n-1)^2} \right)^k \|\nabla_g(\Delta_g^k u)\|_p. \end{aligned}$$

We now move to the second part of the proof. We shall prove the sharpness of $C(n, m, p)$ given in (1.4) in the next subsection.

2.2. The sharpness of $C(n, m, p)$. It remains to check the sharpness of the constant $C(n, m, p)$. To do this, we will construct a function u in such a way that $\|\nabla_g^m u\|_p / \|u\|_p$ approximates $C(n, m, p)^{-1}$. Observe from (2.3) that

$$n\omega_n(\sinh F(s))^{n-1} \geq (n-1)s$$

for any $s \geq 0$ and

$$\lim_{s \rightarrow \infty} \frac{n\omega_n(\sinh F(s))^{n-1}}{(n-1)s} = 1.$$

Hence, for any $\varepsilon > 0$, there is s_0 such that

$$(n-1)s \leq n\omega_n(\sinh F(s))^{n-1} \leq (1+\varepsilon)(n-1)s$$

for all $s \geq s_0$. For any $R > s_0$, let us construct a positive, continuous, non-increasing function f_R on $[0, \infty)$ given by

$$f_R(s) = \begin{cases} s_0^{-1/p} & \text{if } s \in (0, s_0) \\ s^{-1/p} & \text{if } s \in [s_0, R) \\ R^{-1/p} \max\{2 - s/R, 0\} & \text{if } s \geq R. \end{cases} \quad (2.11)$$

Then we define two sequences of functions $\{v_{R,i}\}_{i \geq 0}$, $\{g_{R,i}\}_{i \geq 1}$ as follows: first we set $v_{R,0} = f_R$, then we define $g_{R,i+1}$ as the maximal function of $v_{R,i}$, that is

$$g_{R,i+1}(s) = \frac{1}{s} \int_0^s v_{R,i}(t) dt,$$

and then

$$v_{R,i+1}(s) = \int_s^{+\infty} \frac{t g_{R,i+1}(t)}{(n\omega_n(\sinh F(t))^{n-1})^2} dt,$$

for $i = 0, 1, 2, \dots$. Note that $v_{R,i}$ and $g_{R,i}$ are non-increasing functions. We can explicitly compute the function $g_{R,1}$ as follows: When $s < R$ we have

$$g_{R,1}(s) = \begin{cases} s_0^{-1/p} & \text{if } s \in (0, s_0) \\ p' s^{-1/p} - s_0^{1-1/p} / ((p-1)s) & \text{if } s \in [s_0, R), \end{cases}$$

while for $s \in [R, 2R)$ we have

$$g_{R,1}(s) = \left(\left(p' - \frac{3}{2} \right) R^{1-1/p} - \frac{s_0^{1-1/p}}{p-1} \right) \frac{1}{s} + 2R^{-1/p} - \frac{R^{-1-1/p}s}{2}, \quad \forall s \in [R, 2R),$$

and finally when $s \geq 2R$ we have

$$g_R(s) = \left(p' R^{1-1/p} - \frac{s_0^{1-1/p}}{p-1} \right) \frac{1}{s} + \frac{R^{1-1/p}}{2s}, \quad \forall s \geq 2R.$$

Note that

$$\int_R^{+\infty} g_R(s)^p ds \leq C$$

for some constant $C > 0$ independent of R

In the sequel, we use C to denote various constants which are independent of R and whose values can change from line to line and even in one line if no confusion occurs. We will need the following result.

Proposition 2.1. *For any $i \geq 1$, there exist functions $h_{R,i}$ and $w_{R,i}$ such that $v_{R,i} = h_{R,i} + w_{R,i}$, $\int_0^{+\infty} |w_{R,i}|^p ds \leq C$ and*

$$\frac{1}{(1+\varepsilon)^{2i}} \left(\frac{pp'}{(n-1)^2} \right)^i f_R \leq h_{R,i} \leq \left(\frac{pp'}{(n-1)^2} \right)^i f_R.$$

Proof. Let us define the operator T acting on functions v on $[0, +\infty)$ by

$$Tv(s) = \int_s^{+\infty} \frac{r}{(n\omega_n(\sinh F(r))^{n-1})^2} \left(\frac{1}{r} \int_0^r v(t) dt \right) dr.$$

For simplicity, for each function v on $[0, +\infty)$ we define an associated function \overline{v} on \mathbb{H}^n by

$$\overline{v}(x) = v(|B(0, d(0, x))|).$$

With these notation, it is not hard to see that

$$\|\overline{w_{R,i}}\|_p = \left(\int_0^{+\infty} |w_{R,i}(s)|^p ds \right)^{1/p}$$

for any $i \geq 1$ and

$$-\Delta_g \overline{Tw_{R,i}}(x) = \overline{w_{R,i}}(x)$$

for any $x \in \mathbb{H}^n$. Hence, by the Poincaré inequality, we have

$$\int_0^{+\infty} |Tw_{R,i}(s)|^p ds = \|\overline{Tw_{R,i}}\|_p^p \leq C \|\overline{w_{R,i}}\|_p^p = C \left(\int_0^{+\infty} |w_{R,i}(s)|^p ds \right)^{1/p}.$$

Thus, using an induction argument, it is enough to prove this proposition for $i = 1$. We will perform several explicit estimation for the function $v_{R,1}$. Note that for $s \geq s_0$ we have

$$(n-1)s \leq n\omega_n(\sinh F(s))^{n-1} \leq (1+\varepsilon)(n-1)s.$$

Estimate of $v_{R,1}$ when $s \geq 2R$. Clearly for $s \geq 2R$, we have

$$\begin{aligned} v_{R,1}(s) &\leq \frac{1}{(n-1)^2} \int_s^{+\infty} \frac{(p'+1/2)R^{1-1/p} - s_0^{1-1/p}/(p-1)}{t^2} dt \\ &= \frac{1}{(n-1)^2} \frac{(p'+1/2)R^{1-1/p} - s_0^{1-1/p}/(p-1)}{s}, \end{aligned}$$

and similarly we have

$$v_{R,1}(s) \geq \frac{1}{(1+\varepsilon)^2(n-1)^2} \frac{(p'+1/2)R^{1-1/p} - s_0^{1-1/p}/(p-1)}{s}.$$

Thus an easy calculation shows that

$$\int_{2R}^{+\infty} v_{R,1}(s)^p ds \leq C. \quad (2.12)$$

Estimate of $v_{R,1}$ when $R \leq s < 2R$. For $s \in [R, 2R)$, we first write

$$v_{R,1}(s) = v_{R,1}(2R) + \int_s^{2R} \frac{tg_{R,1}(t)}{(n\omega_n(\sinh F(t))^{n-1})^2} dt.$$

Then we can estimate

$$\begin{aligned} v_{R,1}(2R) + \frac{1}{(1+\varepsilon)^2(n-1)^2} \int_s^{2R} \frac{g_{R,1}(t)}{t} dt \\ \leq v_{R,1}(s) \leq v_{R,1}(2R) + \frac{1}{(n-1)^2} \int_s^{2R} \frac{g_{R,1}(t)}{t} dt. \end{aligned}$$

Note that $v_{R,1}(2R)$ is equivalent to $R^{-1/p}$ and

$$\begin{aligned} \int_s^{2R} \frac{g_{R,1}(t)}{t} dt = \left(\left(p' - \frac{3}{2} \right) R^{1-1/p} - \frac{s_0^{1-1/p}}{p-1} \right) \left(\frac{1}{s} - \frac{1}{2R} \right) \\ + 2R^{-1/p} \ln \frac{2R}{s} - \frac{R^{-1-1/p}(2R-s)}{2}. \end{aligned}$$

This shows that

$$\int_R^{2R} v_{R,1}(s)^p ds \leq C, \quad (2.13)$$

and that $v_{R,1}(R)$ is equivalent to $R^{-1/p}$. Combining the estimates (2.12) and (2.13) gives $\int_R^{+\infty} v_{R,1}(s)^p ds \leq C$.

Estimate of $v_{R,1}$ when $s_0 \leq s < R$. For $s \in [s_0, R)$, we also write

$$v_{R,1}(s) = v_{R,1}(R) + \int_s^R \frac{tg_{R,1}(t)}{(n\omega_n(\sinh F(t))^{n-1})^2} dt.$$

Thus

$$v_{R,1}(R) + \frac{1}{(1+\varepsilon)^2(n-1)^2} \int_s^R \frac{g_{R,1}(t)}{t} dt \leq v_{R,1}(s) \leq v_{R,1}(R) + \frac{1}{(n-1)^2} \int_s^R \frac{g_{R,1}(t)}{t} dt.$$

A simple computation gives

$$\int_s^R \frac{g_{R,1}(t)}{t} dt = pp'(s^{-1/p} - R^{-1/p}) - \frac{s_0^{1-1/p}}{p-1} \left(\frac{1}{s} - \frac{1}{R} \right),$$

which implies that

$$\int_{s_0}^R \left| \int_s^R \frac{g_{R,1}(t)}{t} dt - \frac{pp'}{s^{1/p}} \right|^p ds \leq C.$$

Estimate of $v_{R,1}$ when $s < s_0$. For $s \in (0, s_0)$ we write

$$v_{R,1}(s) = v_{R,1}(s_0) + \int_s^{s_0} \frac{ts_0^{-1/p}}{(n\omega_n(\sinh F(t))^{n-1})^2} dt,$$

therefore

$$|v_{R,1}(s)| \leq C(R^{-1/p} + s_0^{2/n-1/p}).$$

Consequently, we can write $v_{R,1} = h_{R,1} + w_{R,1}$ with $\int_0^{+\infty} |w_{R,1}|^p ds \leq C$ for some constant C independent of R and

$$\frac{1}{(1+\varepsilon)^2} \frac{pp'}{(n-1)^2} f_R \leq h_{R,1} \leq \frac{pp'}{(n-1)^2} f_R.$$

(The way to see this is as follows: Since $\int_R^{+\infty} v_{R,1}(s)^p ds \leq C$, we can choose $h_{R,1} = pp'(n-1)^{-2} f_R$ when $r \geq R$. When $r < s_0$, we choose the same function for $h_{R,1}$. When $s_0 \leq r < R$, we choose $h_{R,1}(s) = pp'(n-1)^{-2} / s^{1/p}$ with a remark that $f_R(s) = s^{-1/p}$ in this scenario.) This finishes our proof of the proposition. \square

We are now in a position to confirm the sharpness of $C(n, m, p)$. For clarity, we split our proof into several small steps.

2.2.1. *The sharpness of $C(n, 1, p)$.* We set $u_R(x) = f_R(|B(0, d(0, x))|)$. It is not hard to see that $u_R \in W^{1,p}(\mathbb{H}^n)$. We also consider the function k_R defined by

$$k_R(\varphi(s))\varphi'(s) = f'_R(s).$$

To finish our proof, we shall test the inequality with the function u_R . First, we use (2.2) to get

$$\int_{\mathbb{H}^n} u_R(x)^p dV_g = \int_0^{+\infty} f_R(s)^p ds = 1 + \ln R - \ln s_0 + \int_0^1 (1-s)^p ds.$$

For the gradient term, we observe that

$$\begin{aligned} \int_{\mathbb{H}^n} |\nabla_g u_R(x)|^p dV_g &= \int_0^{+\infty} k_R(s)^p ds \\ &= - \int_0^{+\infty} k_R(\varphi(s))^p \varphi'(s) ds \\ &= \int_0^{+\infty} (f'_R(s))^p (-\varphi'(s))^{1-p} ds \\ &\leq \frac{(n-1)^p}{p^p} (1+\varepsilon)^p \int_{s_0}^R s^{-1} ds + (n-1)^p (1+\varepsilon)^p R^{-p-1} \int_R^{2R} s^p ds \\ &= \frac{(n-1)^p}{p^p} (1+\varepsilon)^p (\ln R - \ln s_0) + (n-1)^p (1+\varepsilon)^p \int_0^1 (1+s)^p ds. \end{aligned}$$

Hence

$$\inf_{u \in W_0^{1,p}(\mathbb{H}^n) \setminus \{0\}} \frac{\int_{\mathbb{H}^n} |\nabla_g u|^p dV_g}{\int_{\mathbb{H}^n} |u|^p dV_g} \leq \liminf_{R \rightarrow +\infty} \frac{\int_{\mathbb{H}^n} |\nabla_g u_R|^p dV_g}{\int_{\mathbb{H}^n} |u_R|^p dV_g} \leq \frac{(n-1)^p}{p^p} (1+\varepsilon)^p.$$

Since $\varepsilon > 0$ is arbitrary, we obtain

$$\inf_{u \in W_0^{1,p}(\mathbb{H}^n) \setminus \{0\}} \frac{\int_{\mathbb{H}^n} |\nabla_g u|^p dV_g}{\int_{\mathbb{H}^n} |u|^p dV_g} \leq \left(\frac{n-1}{p} \right)^p.$$

Hence the preceding inequality becomes equality. This proves the sharpness of $C(n, 1, p)$. Next, we move to a proof for the sharpness of $C(n, 2, p)$.

2.2.2. *The sharpness of $C(n, 2, p)$.* In this case, we set $u_R(x) = v_{R,1}(|B(0, d(0, x))|)$, then we have

$$-\Delta_g u_R(x) = f_R(|B(0, d(0, x))|).$$

Using this fact and (2.2), we easily obtain

$$\int_{\mathbb{H}^n} |\Delta_g u_R|^p dV_g = \int_0^{+\infty} f_R(s)^p ds = 1 + \ln(R/s_0) + \int_0^1 (1-s)^p ds. \quad (2.14)$$

By Proposition 2.1, we have

$$\begin{aligned} \|u_R\|_p &= \left(\int_0^{+\infty} v_{R,1}(s)^p ds \right)^{1/p} \\ &\geq \left(\int_0^{+\infty} h_{R,1}(s)^p ds \right)^{1/p} - \left(\int_0^{+\infty} |w_{R,1}|^p ds \right)^{1/p} \\ &\geq \frac{1}{(1+\varepsilon)^2} \frac{pp'}{(n-1)^2} \left(\int_0^{+\infty} f_R(s)^p ds \right)^{1/p} - C \\ &= \frac{1}{(1+\varepsilon)^2} \frac{pp'}{(n-1)^2} \left(1 + \ln(R/s_0) + \int_0^1 (1-t)^p dt \right)^{1/p} - C. \end{aligned}$$

Combing this estimate and (2.14) gives

$$C(n, 2, p) \geq \liminf_{R \rightarrow +\infty} \frac{\|u_R\|_p}{\|\Delta_g u_R\|_p} \geq \frac{1}{(1+\varepsilon)^2} \frac{pp'}{(n-1)^2}.$$

Since $\varepsilon > 0$ is arbitrary, we conclude that

$$C(n, 2, p) \geq \frac{pp'}{(n-1)^2}$$

and this finishes our proof for the case $m = 2$.

2.2.3. *The sharpness of $C(n, 2k, p)$ with $k \geq 2$.* In this case, we set

$$u_R(x) = v_{R,k}(|B(0, d(0, x))|),$$

then it is clear to see that

$$(-\Delta_g)^k u_R(x) = f_R(|B(0, d(0, x))|).$$

By Proposition 2.1, we can write $v_{R,k} = h_{R,k} + w_{R,k}$ with $\int_0^{+\infty} |w_{R,k}|^p ds \leq C$ and

$$\frac{1}{(1+\varepsilon)^{2k}} \left(\frac{pp'}{(n-1)^2} \right)^k f_R \leq h_{R,k} \leq \left(\frac{pp'}{(n-1)^2} \right)^k f_R.$$

Using again the argument in proving the sharpness of $C(n, 2, p)$, we obtain the sharpness of $C(n, 2k, p)$.

2.2.4. *The sharpness of $C(n, 2k+1, p)$ with $k \geq 1$.* In the previous argument, we can find a function u_R on \mathbb{H}^n such that

$$(-\Delta_g)^k u_R(x) = f_R(|B(0, d(0, x))|)$$

and that

$$\|u_R\|_p \geq \frac{1}{(1+\varepsilon)^{2k}} \left(\frac{pp'}{(n-1)^2} \right)^k \left(\int_0^{+\infty} f_R(s)^p ds \right)^{1/p} - C.$$

From the proof of the sharpness of $C(n, 1, p)$, we know that

$$\int_{\mathbb{H}^n} |\nabla_g(\Delta_g^k u_R)|^p dV_g \leq \left(\frac{n-1}{p} \right)^p (1+\varepsilon)^p \ln \frac{R}{s_0} + (n-1)^p (1+\varepsilon)^p \int_0^1 (1-t)^p dt.$$

Combining these two estimate implies the sharpness of $C(n, 2k+1, p)$.

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